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On a Class of Generalized Gonćarov Polynomials

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The convergence properties of the class of Gonćarov polynomials $\{Q_k(z; z_0, \ldots, z_{k-1})\}$ generated through the qth derivative ($q \neq 1$) are investigated in the present paper when $z_k = a\,t^k$, $k \geqslant 0$, where a and t are any complex numbers. The investigations carried on here cover the possible ranges of variation of t and q, namely, $|t| >, =, < 1$ when $0 \leqslant q < 1$, and $|t| >, =, < 1/q$ when $q > 1$. Except for the cases $|t| > 1$, $q < 1$, and $|t| > 1/q$, $q > 1$, the results obtained in the present work ensure the effectiveness of the set $\{Q_k(z)\}$ in finite circles.

1. INTRODUCTION

Let $(z_k)_0^\infty$ be a sequence of given complex numbers; the *Gonćarov set* $\{G_k(z; z_0, \ldots, z_{k-1})\}$, associated with the points (z_k), is defined by Gonćarov [5] to the simple[1] set of polynomials given by

$$G_0(z) = 1,$$

$$G_k(z) = G_k(z; z_0, \ldots, z_{k-1}) = \int_{z_0}^{z} ds_1 \int_{z_1}^{s_1} ds_2 \cdots \int_{z_{k-1}}^{s_{k-1}} ds_k \qquad (k \geqslant 1).$$

These polynomials "generate" any function $f(z)$ regular in a compact set containing the points (z_k) through the Gonćarov series

$$f(z) \sim \sum_{k=0}^{\infty} f^{(k)}(z_k)\, G_k(z; z_0, \ldots, z_{k-1}). \tag{1.1}$$

The Gonćarov polynomials have been the interest of many authors of whom we may mention Macintyre [6], the author [7], and Abd-el-Monem

[1] The reader is supposed to be acquainted with the theory of basic sets of polynomials as given by Whittaker [8].

and the author [1]. Putting $f(z) = z^n$ in (1.1) we obtain the following constructive relations for the Gonćarov set

$$G_0(z) = 1, \tag{1.2}$$
$$z^n = \sum_{k=0}^{n} (n!/(n-k)!)\, z_k^{n-k} G_k(z; z_0, \ldots, z_{k-1}) \qquad (n \geqslant 1).$$

Recently, Buckholtz and Frank[2] [3] introduced and extensively studied a generalization of Gonćarov polynomials which can be described as follows.

Let $(d_n)_1^\infty$ denote a nondecreasing sequence of positive numbers and let D denote the operator which transforms the function $f(z) = \sum_{n=0}^\infty a_n z^n$, regular at the origin, to $Df(z) = \sum_{n=1}^\infty d_n a_n z^{n-1}$. Writing $e_0 = 1$, $e_n = (d_1 d_2 \cdots d_n)^{-1}$, $n \geqslant 1$, the generalized Gonćarov polynomials $\{P_k(z)\}$, associated with the sequence of points $(z_k)_0^\infty$, are defined by the relations

$$P_0(z) = 1, \tag{1.3}$$
$$e_n z^n = \sum_{k=0}^{n} e_{n-k} z_k^{n-k} P_k(z; z_0, \ldots, z_{k-1}) \qquad (n \geqslant 1).$$

When $d_n = n$, the relations (1.3) reduce to the form (1.2) and hence the polynomials $\{P_k(z)\}$ will then reduce to the proper Gonćarov polynomials $\{G_k(z)\}$.

In the present paper we propose to investigate the convergence properties of a certain class of the generalized Gonćarov polynomials, namely the polynomials $\{Q_k(z)\}$ for which the operator D is the qth *derivative operator* Dq, defined by[3]

$$D_q - f(z) = (f(qz) - f(z))/z(q-1) = \sum_{n=1}^{\infty} [n]\, a_n z^{n-1}, \tag{1.4}$$

when $f(z) = \sum_{n=0}^\infty a_n z^n$, where q is a nonnegative[4] number different from 1 and $[n]$, called the q-analog of the integer n, equal to $(q^n - 1)/(q - 1)$. Thus the class $\{Q_k(z)\}$ of generalized Gonćarov polynomials to be studied here, is constructed as follows.

$$Q_0(z) = 1, \tag{1.5}$$
$$z^n = \sum_{k=0}^{n} ([n]!/[n-k]!)\, z_k^{n-k} Q_k(z; z_0, \ldots, z_{k-1}) \qquad (n \geqslant 1),$$

[2] Throughout this work we shall refer to these authors by B. and F.

[3] For definition and account on the q-derivative and the q-analogs $[n]$, $[n]!$, $[{n \atop k}]$, cf, e.g., [2; pp. 616–618] .

[4] When $q = 0$, then D_0 is the shift operator ζ which transforms $f(z)$ to $\zeta f(z) = \sum_{n=1}^\infty a_n z^{n-1}$ (cf. [3, p. 350; 11.4–6]) and the corresponding Gonćarov polynomials are the polynomials $\{B_n(z)\}$, given by $z^n = \sum_{k=0}^n z_k^{n-k} B_k(z; z_0, \ldots, z_{k-1})$.

where $[k]! = [1][2] \cdots [k]$, is the q-analog of the factorial $k!$; also, the Gonćarov series (1.1) will now be

$$f(z) \sim \sum_{k=0}^{\infty} D_q^k f(z_k)\, Q_k(z; z_0, \ldots, z_{k-1}). \tag{1.6}$$

Now, since the study of B. and F. was primarily concerned with the case where the points $(z_k)_0^\infty$ lie on the unit circle, in order to obtain results of a new character, we consider here the case where

$$z_k = at^k \qquad (k \geqslant 0),$$

where a and t are any complex numbers; thus the Gonćarov polynomials considered here are

$$Q_k(z; a, at, \ldots, at^{k-1}) \qquad (k \geqslant 1). \tag{1.7}$$

As in the case of proper Gonćarov polynomials (cf. [1, 7]), the results obtained in the present work are conclusive, and we write

$$|a| = \alpha, \qquad |t| = \beta,$$

and consider the function

$$\phi(z; t) = \sum_{n=0}^{\infty} t^{-(1/2)n(n-1)} z^n/[n]!. \tag{1.8}$$

When $0 \leqslant q < 1$ and $\beta > 1$ or when $q > 1$ and $q\beta > 1$, this function is an entire function of order 0. On the other hand, when $0 \leqslant q < 1$ and $\beta = 1$ or when $q > 1$ and $q\beta = 1$, the function has the circle $|z| = 1/(1-q)$ or $|z| = q/(q-1)$, respectively, as its circle of regularity. If, in the latter cases, the function $\phi(z; t)$ has zeros inside its circle of regularity, we define the numbers σ and τ as follows.

$$\sigma = \min[|z|: \phi(z; t) = 0] \qquad (0 \leqslant q < 1; \beta = 1), \tag{1.9}$$

$$\tau = \min[|z|: \phi(z; t) = 0] \qquad (q > 1; q\beta = 1), \tag{1.10}$$

so that

$$0 < \sigma < 1/(1-q); \qquad 0 < \tau < q/(q-1). \tag{1.11}$$

The main results of the present work are formulated in the following theorems.

THEOREM 1.1. *Suppose that* $0 \leqslant q < 1$. *When* $\beta > 1$, *the Gonćarov* *set $\{Q_k(z; a, at,..., at^{k-1})\}$ will be of infinite order, and when $\beta = 1$, the Gončarov set will be effective in $|z| \leqslant r$, if and only if, $r \geqslant r_1$, where $r_1 = \alpha/\sigma(1-q)$ or $r_1 = \alpha$ according to whether the function $\phi(z; t)$ has zeros inside its circle of regularity $|z| = 1/(1-q)$ or not.*

THEOREM 1.2. *Suppose that $q > 1$. If $q\beta > 1$, the above Gončarov set will be of infinite order, and if $q\beta = 1$, the Gončarov set will be effective in $|z| \leqslant r$, if and only if $r \geqslant r_2$, where $r_2 = \alpha q/\tau(q-1)$ or $r_2 = \alpha$ according to whether the function $\phi(z; t)$ has zeros inside its circle of regularity $|z| = q/(q-1)$ or not.*

The cases of existence or nonexistence of zeros of the function $\phi(z; t)$ inside its circle of regularity are exemplified and lead to Theorems 3.1, 3.2, and 3.3 below. Also, the ranges of variation of q and β, hitherto uncovered by Theorems 1.1 and 1.2, namely,

$$0 \leqslant q < 1 \quad \text{and} \quad \beta < 1; \qquad \text{and} \qquad q > 1 \quad \text{and} \quad q\beta < 1,$$

need special treatment in Sections 4 and 5 below. The results obtained are given in the following theorem.

THEOREM 1.3. *When $0 \leqslant q < 1$ and $\beta < 1$, or when $q > 1$ and $q\beta < 1$, the Gončarov set $\{Q_k(z; a, at,..., at^{k-1})\}$ will be effective in $|z| \leqslant r$, if and only if, $r \geqslant \alpha$.*

2. PROOFS OF THEOREMS 1.1 AND 1.2

These theorems cover the following ranges of variation of q and t

$$0 \leqslant q < 1, \quad \beta \geqslant 1 \qquad \text{and} \qquad q > 1, \quad q\beta \geqslant 1, \tag{2.1}$$

and the results are derived through the function $\phi(z; t)$, of (1.8), whose existence is ensured over the above ranges. In fact, writing

$$h_k(z_0, ..., z_{k-1}) = Q_k(0; z_0, ..., z_{k-1}) \qquad (k \geqslant 1), \tag{2.2}$$

for any sequence $(z_k)_0^\infty$, we adopt the notation

$$u_0 = 1; \qquad u_k = t^{-(1/2)k(k-1)} h_k(1, t, ..., t^{k-1}) \qquad (k \geqslant 1). \tag{2.3}$$

Putting in the identity of B. and F. [3; *p.* 357],

$$h_n(z_0, ..., z_{n-1}) = -\sum_{k=0}^{n-1} (z_k^{n-k}/[n-k]!)\, h_k(z_0, ..., z_{k-1}),$$

$z_k = at^k$, $k \geqslant 0$, and applying (2.3), we obtain

$$u_0 = 1, \qquad \sum_{k=0}^{n} (t^{-(1/2)k(k-1)}/[k]!)\, u_{n-k} = 0 \qquad (n \geqslant 1).$$

That is to say,

$$\sum_{k=0}^{\infty} u_k z^k = 1/\phi(z; t). \tag{2.4}$$

We need also the following relation which is deducible from [3, formula (2.9)].

$$Q_n(z; a, at,..., at^{n-1}) = \sum_{k=0}^{n} (a^k t^{(1/2)k(2n-k-1)}/[n-k]!)\, u_k z^{n-k}. \tag{2.5}$$

We now assume that $0 \leqslant q < 1$, $\beta \geqslant 1$ and proceed to prove Theorem 1.1. When $\beta > 1$, the function $\phi(z; t)$ is an entire function of zero order and therefore it must have zeros in the finite part of the plane. Let

$$\rho = \min[|\, z \,|: \phi(z; t) = 0] < \infty;$$

then by (2.4) we should have

$$\limsup_{n\to\infty} |\, u_n \,|^{1/n} = 1/\rho > 0. \tag{2.6}$$

Now, adopting the notation

$$M_k(r) = \sup_{|z| \leqslant r} |\, Q_k(z; a, at,..., at^{k-1})| \qquad (k \geqslant 1), \tag{2.7}$$

then (1.5) and (2.5) imply that

$$\begin{aligned} \omega_n(r) &= \sum_{k=0}^{n} ([n]!/[n-k]!)\, \alpha^{n-k} \beta^{k(n-k)} M_k(r) \\ &\geqslant [n]!\, \alpha^n \beta^{(1/2)n(n-1)} \,|\, u_n \,|, \end{aligned} \tag{2.8}$$

where $\omega_n(r)$ is the Cannon sum for the Gonćarov set $\{Q_n(z)\}$. It easily follows from (2.6) and (2.8) that the set $\{Q_n(z)\}$ is of infinite order.

To complete the proof of the theorem we suppose that $\beta = 1$; in this case the function $\phi(z; t)$ is regular in $|\, z \,| < 1/(1-q)$. Assuming that this function has zeros in its circle of regularity, define σ as in (1.9). Hence, the Cannon function for the set $\{Q_n(z)\}$ will, in view of (2.7), be

$$\lambda(r) = \limsup_{n\to\infty} \{\omega_n(r)\}^{1/n} \geqslant \alpha/\sigma(1-q) = r_1\,.$$

Moreover, if σ_1 is any positive number less than σ, then (2.4) yields[5]

$$|u_k| < K/\sigma_1^k \qquad (k \geqslant 0). \tag{2.9}$$

Hence, if $r \geqslant r_1$, the Cannon function $\lambda(r)$ for the set can be evaluated from the fact that $\lim_{n\to\infty}\{1/[n]!\}^{1/n} = 1 - q$ and from (2.5), (2.8), and (2.9). Thus we get

$$\lambda(r) \leqslant (\sigma/\sigma_1)\{(1 - q_1)/(1 - q)\}\, r,$$

where q_1 is an arbitrary positive number less than q. We thus conclude that

$$\lambda(r) = r_1 \quad \text{for} \quad 0 < r \leqslant r_1 \qquad \text{and} \qquad \lambda(r) = r \quad \text{for} \quad r \geqslant r_1,$$

and that the set $\{Q_n(z)\}$ is effective in $|z| \leqslant r$ if and only if $r \geqslant r_1$, where r_1 is equal to $\alpha/\sigma(1 - q)$.

We now assume that the function $\phi(z; t)$ has no zeros in $|z| < 1/(1 - q)$; then it can be verified from (2.4) that

$$\limsup_{k\to\infty} |u_k|^{1/k} \leqslant 1 - q. \tag{2.10}$$

Following the same lines as above, the Cannon function can be obtained for $r \geqslant \alpha$, through the relation (2.10). Thus we get

$$\lambda(r) = r \qquad (r \geqslant \alpha). \tag{2.11}$$

Since, however, we always have $\lambda(r) \geqslant \alpha$, $r > 0$, in veiw of (2.8), we can deduce from (2.11) that the set $\{Q_n(z)\}$ is effective in $|z| \leqslant r$ if and only if $r \geqslant r_1$, where r_1 is here equal to α. Theorem 1.1 is therefore established.

Theorem 1.2 can be proved through a procedure similar to that followed in the proof of Theorem 1.1 with obvious changes. The proof of this theorem is therefore omitted.

3. Examples

The existence and nonexistence of zeros of the function $\phi(z; t)$ in its circle of regularity are exemplified in the present section.

When $0 \leqslant q < 1$, $\beta = 1$, the "no-zero" case is illustrated by setting $t = 1$ and it leads to the following familiar result.

THEOREM 3.1. *The set* $\{Q_k(z)\}$ *of interpolation polynomials, given by*

[5] K denotes positive finite number, independent of k, which does not retain the same values at different occurrences.

$Q_0(z) = 1$, $Q_k(z) = (z - a)(z - aq) \cdots (z - aq^{k-1})$ $(q < 1; k \geqslant 1)$, *is effective in* $|z| \leqslant r$, *if and only if*, $r \geqslant \alpha$.

Proof. Putting $t = 1$, we obtain the function

$$\phi(z; 1) = E(z) = \sum_{n=0}^{\infty} z^n/[n]!, \tag{3.1}$$

which, according to [4; pp. 195–196], has no zeros inside the circle of regularity $|z| = 1/(1 - q)$. Hence, by Theorem 1.1 above, the corresponding Conćarov set $\{Q_k(z; a, a,..., a)\}$ will be effective in $|z| \leqslant r$ if and only if $r \geqslant \alpha$. To show that

$$[k]!\, Q_k(z; a, a,..., a) = (z - a)(z - aq) \cdots (z - aq^{k-1}) \qquad (k \geqslant 1), \tag{3.2}$$

we consider the generating function

$$G(z, \omega) = \sum_{k=0}^{\infty} Q_k(z; a, a,..., a)\, \omega^k.$$

Applying the aforementioned reference [4], it can be shown, by (2.4) and (2.5), that

$$G(z; \omega) = \prod_{j=0}^{\infty} \left\{ \frac{1 - a\omega q^j(1 - q)}{1 - z\omega q^j(1 - q)} \right\}$$

$$= \sum_{k=0}^{\infty} \frac{(z - a)(z - aq) \cdots (z - aq^{k-1})\, \omega^k}{[k]!},$$

from which (3.2) follows at once.

When $q = 0$, the result is otherwise obvious, since the corresponding set of polynomials

$$B_k(z; a, a,..., a) = (z - a)\, z^{k-1} \qquad (k \geqslant 1),$$

is effective in $|z| \leqslant r$ if and only if $r \geqslant \alpha$.

The existence of zeros of the function[6] $\phi(z; t)$, $0 < q < 1$, $\beta = 1$, in its circle of regularity, $|z| = 1/(1 - q)$, is exemplified by taking $t = -1$, and the resulting "two-point" problem is governed by the following theorem.

[6] When $q = 0$, the function $\phi(z; -1) = \sum_{n=0}^{\infty} (-1)^{-(1/2)n(n-1)} z^n$ is equal to $(z + 1) \times (1 + z^2)^{-1}$, which has its zero on its circle of regularity.

THEOREM 3.2. *Let $f(z)$ be an analytic function which is regular in $|z| \leqslant \alpha/\xi$, where ξ is the root of the equation*

$$h(x; q) \equiv \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)(1-q^{2k+1})} = \pi/4 \qquad 0 < \xi < 1. \tag{3.3}$$

Suppose further that $D_q^n f\{(-1)^n a\} = 0$, $n \geqslant 0$; then $f(z) \equiv 0$.

Proof. Setting $t = -1$, we get

$$\phi(z; -1) = \tfrac{1}{2}\{(1-i)\, E(iz) + (1+i)\, E(-iz)\} \qquad (|z| < 1/(1-q)),$$

where $E(z)$ is the function (3.1). Thus a zero of $\phi(z; -1)$ in $|z| < 1/(1-q)$ is given by

$$E(iz)/E(-iz) = -i. \tag{3.4}$$

Putting

$$z = u/(1-q) \qquad (|u| < 1), \tag{3.5}$$

Eq. (3.4) may be put in the form

$$\prod_{j=0}^{\infty} \left(\frac{1 + iuq^j}{1 - iuq^j}\right) = -i,$$

or, taking logarithms,

$$\sum_{k=0}^{\infty} \frac{(-1)^k u^{2k+1}}{(2k+1)(1-q^{2k+1})} = -\pi/4,$$

which yields (3.3) when x is written for $-u$.

To show that Eq. (3.3) has a root ξ lying between 0 and 1, we observe that $h(1; 0) = \pi/4$, and, when $q > 0$, that

$$h(x; q) \geqslant h(x; 0) + \sum_{k=0}^{\infty} \frac{2(xq)^{4k+1}}{(4k+1)(4k+3)} \qquad (0 \leqslant x \leqslant 1).$$

It follows that

$$h(1; q) \geqslant h(1; 0) + 2q/3 > \pi/4.$$

Since $h(0; q) = 0$, it follows, from the continuity and monotony of $h(x; q)$ in $0 \leqslant x < 1$, that $h(\xi; q) = \pi/4$ for $0 < \xi < 1$.

Applying Theorem 1.1 to the foregoing Gončarov set $\{Q_k(z; a, -a, \ldots, (-1)^{k-1} a)\}$, we deduce that this set will be effective in $|z| \leqslant \alpha/\xi$, in view of (3.5). In this case the basic series (1.6) for the prescribed function $f(z)$

of the theorem identically vanishes. Hence $f(z) \equiv 0$ and the theorem is proved.

Now, when $q > 1$, $q\beta = 1$, the results are quite similar. Thus if $t = 1/q$ the resulting set is

$$[k]!\, Q_k(z; a, a/q, ..., a/q^{k-1}) = (z - a)(z - aq^{-1}) \cdots (z - aq^{-k+1}),$$

which is effective in $|z| \leqslant r$ if and only if $r \geqslant \alpha$. Also, if we put $t = -1/q$, a result similar to that of Theorem 3.2 is obtained; it is stated as follows.

THEOREM 3.3. *Let $f(z)$ be an analytic function which is regular in $|z| \leqslant \alpha/\xi$, where ξ is the root of the equation*

$$h(x; u) \equiv \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)(1-u^{2k+1})} = \pi/4, \qquad u = 1/q.$$

If[7] *$D_q^n f\{(-1)^n a/q^n\} = 0$, $n \geqslant 0$, then $f(z) \equiv 0$.*

Finally, we observe that, when we take $t = e^{i\theta}$, $q < 1$, then (2.3) and (2.4) imply that

$$\limsup_{n\to\infty} |h_n(1, e^{i\theta}, ..., e^{i(n-1)\theta})|^{1/n} = 1/\sigma(\theta),$$

where $\sigma(\theta)$ is the modulus of a zero of least modulus of the function $\phi(z; \theta) = \sum_{n=0}^{\infty} e^{-in(n-1)\theta} \cdot z^n/[n]!$ (if this zero exists) in $|z| < 1/(1-q)$. Applying therefore the result of B. and F. about the Whittaker constant W [3; p. 351], we conclude that, corresponding to the operator D_q, $q < 1$, we shall have

$$W(D_q) \leqslant \inf_{0<\theta<2\pi} \sigma(\theta).$$

4. INTRODUCTORY STUDY OF THE CASES: $q < 1$, $\beta < 1$ AND $q > 1$, $q\beta < 1$

The investigation in this and the following section concerns the ranges of variation of t and q hitherto unconsidered, namely,

$$0 \leqslant q < 1, \quad \beta < 1 \qquad \text{and} \qquad q > 1, \quad \beta < 1/q. \tag{4.1}$$

In These ranges of variation of q and t, the function $\phi(z; t)$, through which the study has been hitherto made, is undefined and a different procedure has to be followed. Following the pattern of study carried out for the proper Gonćarov polynomials (cf. [1]), it is proposed to establish here an identity

[7] It should be observed that, when $q > 1$ and $f(z)$ is regular in $|z| \leqslant r$, then $D_q f(z)$ will be regular in $|z| \leqslant r/q$.

analogous to Euler's formula for homogeneous functions. Application of this identity, which is actually obtained in Lemma 4.3 below, leads to the construction, in Theorem 5.1 below, of an explicit upper bound for the maximum modulus $M_n(r)$, given by (2.7). The proof of Theorem 1.3 above is a mere application of this bound. We first establish the following two introductory lemmas.

LEMMA[8] 4.1. *For $k \geqslant 1$, we have*

$$Q_k(z; z_0, ..., z_{k-1}) = h_k(z_0, z_1, ..., z_{k-1}) - h_k(z, z_1, ..., z_{k-1}). \tag{4.2}$$

Proof. For $k = 1$, we have, from (1.5),

$$Q_1(z; z_0) = z - z_0 = h_1(z_0) - h_1(z),$$

so that (4.2) is true for $k = 1$. Suppose that (4.2) is satisfied for $k = 1, 2, ..., n - 1$. Hence (1.5) and (4.2) imply that

$$\begin{aligned} Q_n(z; z_0, z_1, ..., z_{n-1}) &= \frac{z^n}{[n]!} + \sum_{k=1}^{n-1} \frac{z_k^{n-k}}{[n-k]!} h_k(z, z_1, ..., z_{k-1}) \\ &\quad - \frac{z_0{}^n}{[n]!} - \sum_{k=1}^{n-1} \frac{z_k^{n-k}}{[n-k]!} h_k(z_0, z_1, ..., z_{k-1}) \\ &= h_n(z_0, z_1, ..., z_{n-1}) - h_n(z, z_1, ..., z_{n-1}). \end{aligned}$$

Thus (4.2) is established.

LEMMA 4.2. *When $1 \leqslant k \leqslant n$, the following relation is true.*

$$[D_q^k z Q_n(z; z_1, ..., z_n)]_{z=z_k} = q^k z_k Q_{n-k}(z_k; z_{k+1}, ..., z_n). \tag{4.3}$$

Proof. We first establish the following property of the q-derivative operator D_q

$$D_q^j\{zf)z)\} = zD_q^j f(z) + [j]\, D_q^{j-1} f(qz) \tag{4.4}$$

for any positive integer j, where $f(z) = \sum_{n=0}^{\infty} a_n z^n$, is any function regular about the origin. In fact, if $[n]_j = [n][n-1] \cdots [n-j+1]$, $1 \leqslant j \leqslant n$, then it is seen that

$$[n+1]_j - [n]_j = [j]\, q^{n-j+1} [n]_{j-1}\, . \tag{4.5}$$

[8] Identity (4.2) can be proved for the generalized Gonĉarov polynomials of B. and F.

Hence, we obtain, from Definition (1.4) of D_q and from (4.5),

$$D_q^j\{zf(z)\} = \sum_{n=j-1}^{n} [n+1]_j\, a_n z^{n-j+1}$$

$$= z \sum_{n=j}^{\infty} [n]_j\, a_n z^{n-j} + [j] \sum_{n=j-1}^{\infty} [n]_{j-1}\, a_n (qz)^{n-j+1},$$

which is (4.4). Putting $f(z) = Q_n(z; z_1, ..., z_n)$ into (4.4) we get

$$D_q^k\{zQ_n(z; z_1, ..., z_n)\} = zD_q^k Q_n(z; z_1, ..., z_n) + [k][D_q^{k-1} Q_n(\xi; z_1, ..., z_n)]_{\xi=qz}\,.$$

The B. and F. formula [3, formula (2.6)] is now applied to the above relation; thus we obtain

$$[D_q^k\{zQ_n(z; z_1, ..., z_n)\}]_{z=z_k} = z_k Q_{n-k}(z_k\,; z_{k+1}, ..., z_n) + [k]\, Q_{n-k+1}(qz_k\,; z_k, ..., z_n). \tag{4.6}$$

An appeal to Lemma 4.1 for the polynomial Q_{n-k+1} on the righ-hand side (4.6) readily yields

$$Q_{n-k+1}(qz_k\,; z_k, ..., z_n) = h_{n-k+1}(z_k, z_{k+1}, ..., z_n) - h_{n-k+1}(qz_k, z_{k+1}, ..., z_n) = (q-1)\, z_k Q_{n-k}(z_k\,; z_{k+1}, ..., z_n).$$

Insertion of this relation into (4.6) leads to the required relation (4.3), and the lemma is proved.

The analog of Euler's formula[9] is obtained in the following lemma.

LEMMA 4.3. *The Gonćarov polynomials $\{Q_n(z)\}$ satisfy the following identity.*

$$[n]\, Q_n(z; z_0, ..., z_{n-1}) = zQ_{n-1}(z; z_1, ..., z_{n-1}) - \sum_{k=0}^{n-1} q^k z_k Q_k(z; z_0, ..., z_{k-1}) \cdot Q_{n-k-1}(z_k\,; z_{k+1}, ..., z_{n-1}) \quad (n \geqslant 1). \tag{4.7}$$

Proof. Being a polynomial of degree n in z, $zQ_{n-1}(z; z_1, ..., z_{n-1})$ admits a unique linear representation of the form

$$zQ_{n-1}(z; z_1, ..., z_{n-1}) = a_0 + \sum_{j=1}^{n} a_j Q_j(z; z_0, ..., z_{j-1}). \tag{4.8}$$

[9] Cf. Macintyre [6; p. 242, l. 16, and formulas (2.2), (2.3)].

First, we put $z = z_0$ in (4.8) to obtain

$$a_0 = z_0 Q_{n-1}(z_0 ; z_1 ,..., z_{n-1}). \tag{4.9}$$

Moreover, in view of (1.5) and comparing coefficients of z^n on both sides of (4.8), we deduce that

$$a_n = [n]. \tag{4.10}$$

Finally, to obtain the coefficients $(a_k)_1^{n-1}$, we operate on both sides of (4.8) by D_q^k and put $z = z_k$; thus

$$[D_q^k\{zQ_{n-1}(z; z_1 ,..., z_{n-1})\}]_{z=z_k} = \sum_{j=k}^{n} a_j[D_q^k Q_j(z; z_0 ,..., z_{j-1})]_{z=z_k}.$$

Hence, Lemma 4.2 implies that

$$a_k = q^k z_k Q_{n-k-1}(z_k ; z_{k+1} ,..., z_{n-1}), \tag{4.11}$$

and hence the required identity (4.7) follows from (4.8)–(4.11). Lemma 4.3 is therefore proved.

When $z_k = at^k$, Lemma 4.3 leads to the following recurrence inequality between the maximum moduli $M_k(r)$ of (2.7).

LEMMA 4.4. *When $n \geqslant 2$ and $r \geqslant \alpha$, we have*

$$[n]\, M_n(r) \leqslant (r + \alpha)\, \beta^{n-1} M_{n-1}(r/\beta) + \alpha \sum_{k=1}^{n-1} q^k \beta^{(k+1)(n-k)-1} M_k(r)\, M_{n-k-1}(\alpha/\beta). \tag{4.12}$$

Proof. Putting $z_k = at^k$ in (4.7) we obtain

$$[n]\, Q_n(z; a, at,..., at^{n-1}) = zQ_{n-1}(z; at,..., at^{n-1}) - a \sum_{k=0}^{n-1} q^k t^k Q_k(z; a, at,..., at^{k-1})\, Q_{n-k-1}(at^k; at^{k+1},..., at^{n-1}).$$

Whence, applying B. and F. formula [3; formula (2.5)], it follows that

$$[n]\, Q_n(z; a, at,..., at^{n-1}) = zt^{n-1} Q_{n-1}(z/t; a, at,..., at^{n-2}) - a \sum_{k=0}^{n-1} q^k t^{(k+1)(n-k)-1} Q_k(z; a,..., at^{k-1})\, Q_{n-k-1}(a/t; a,..., at^{n-k-2}). \tag{4.13}$$

The required inequality (4.12) follows from (4.13), when $n \geqslant 2$, on account of the fact that $M_{n-1}(\alpha/\beta) \leqslant M_{n-1}(r/\beta)$, if $r \geqslant \alpha$.

5. Main Result

The main result of the foregoing study covering the ranges (4.1) of variation of q and β, and leading to the proof of Theorem 1.3 above, is an upper estimate of $M_n(r)$ in terms of r, q, and β. In fact, putting

$$\beta^{1/2} = \gamma < 1 \qquad \text{when} \qquad 0 < q < 1, \tag{5.1}$$

and

$$(q\beta)^{1/2} = \eta < 1 \qquad \text{when} \qquad q > 1, \tag{5.2}$$

the following theorem is to be established.

Theorem 5.1. *Suppose that $n \geqslant 2$ and $r \geqslant \alpha$. Then*

(i) *when* $q = 0$, $\beta < 1$,

$$M_n(r) \leqslant \prod_{j=0}^{n-1} (r + \alpha\beta^j); \tag{5.3}$$

(ii) *when* $0 < q < 1$, $\beta < 1$,

$$M_n(r) \leqslant \frac{r+\alpha}{[n]!} \prod_{j=1}^{n-1} (r + 2\alpha j\gamma^j); \tag{5.4}$$

and

(iii) *when* $q > 1$, $q\beta < 1$,

$$M_n(r) \leqslant \frac{r+\alpha}{[n]!} \prod_{j=1}^{n-1} (r + 2\alpha j\eta^j). \tag{5.5}$$

Proof. First of all, for $n = 0, 1$, we have, from Definition (1.5) of the set $\{Q_k(z)\}$,

$$M_0(r) = 1, \qquad M_1(r) = r + \alpha. \tag{5.6}$$

Now, for assertion (i) of the theorem, putting $q = 0$ in (4.12), we get

$$M_n(r) \leqslant (r + \alpha)\, \beta^{n-1} M_{n-1}(r/\beta),$$

which implies (5.3) by reduction.

To establish the remaining assertions of the theorem we observe that easy calculations from (4.12) for $n = 2$ lead to the inequalities

$$M_2(r) \leqslant ((r+\alpha)/[2]!)(r + 2\alpha\gamma) \qquad (0 < q < 1; \beta < 1; r \geqslant \alpha), \tag{5.7}$$

and

$$M_2(r) \leqslant ((r+\alpha)/[2]!)(r+2\alpha\eta) \qquad (q>1;\, q\beta<1;\, r \geqslant \alpha). \tag{5.8}$$

Thus (5.4) and (5.5) are both true for $n = 2$. To proceed further it should be noted that

$$\begin{bmatrix} n \\ k \end{bmatrix} \leqslant \binom{n}{k} \qquad (1 \leqslant k \leqslant n;\, 0<q<1), \tag{5.9}$$

and therefore

$$\begin{bmatrix} n \\ k \end{bmatrix} \leqslant q^{k(n-k)} \binom{n}{k} \qquad (1 \leqslant k \leqslant n;\, q>1). \tag{5.10}$$

Also, a straightforward arithmetic calculation implies that

$$\begin{aligned}(r+\alpha\mu)(r+2\alpha\mu^2) &= (r+2\alpha\mu)(r+4\alpha\mu^2) \\ &\quad - \alpha\mu(r+2\alpha\mu^2) - 2\alpha\mu^2(r+2\alpha\mu),\end{aligned} \tag{5.11}$$

where r, α, μ are positive numbers.

We now suppose that $0<q<1$, $\beta<1$; putting $n = 3$ in (4.12) and applying (5.1), (5.6), (5.7), and (5.9) it can be verified that

$$[3]!\, M_3(r)/(r+\alpha) \leqslant (r+\alpha\gamma)(r+2\alpha\gamma^2) + 2\alpha\gamma^4(\alpha+\alpha\gamma) + \alpha\gamma^4(r+2\alpha\gamma). \tag{5.12}$$

Whence, an appeal to (5.11), with $\mu = \gamma$, gives

$$[3]!\, M_3(r)/(r+\alpha) \leqslant (r+2\alpha\gamma)(r+4\alpha\gamma^2).$$

Thus (5.4) is true for $n = 3$ also; assume the validity of (5.4) for $n = 2, 3, 4,\ldots, m$, where $m \geqslant 3$, write $n = m + 1$ in (4.12) and apply (5.1), (5.4), (5.6), and (5.9) and the fact that $0<q<1$. The following inequality is thus easily arrived at

$$\begin{aligned}[m+1]!\, M_{m+1}(r)/(r+\alpha) &\leqslant (r+\alpha\gamma) \prod_{j=1}^{m-1} (r+2\alpha j\gamma^{j+1}) \\ &+ \alpha^2 \binom{m}{1} \gamma^{2m}(1+\gamma) \prod_{j=1}^{m-2} (\alpha+2\alpha j\gamma^{j+1}) \\ &+ \alpha^2(1+\gamma) \sum_{k=2}^{m-1} \binom{m}{k} \gamma^{2k(m+1-k)} \left\{ \prod_{j=1}^{k-1} (r+2\alpha j\gamma^j) \right\} \\ &\times \left\{ \prod_{j=1}^{m-k-1} (\alpha+2\alpha j\gamma^{j+1}) \right\} + \alpha\gamma^{2m} \prod_{j=1}^{m-1} (r+2\alpha j\gamma^j).\end{aligned} \tag{5.13}$$

We then turn to the case where $q > 1$, $q\beta < 1$. The alternative of (5.12) is obtained through (4.12) with $n = 3$, (5.2), (5.6), (5.8), and (5.10), in the form

$$[3]!\, M_3(r)/(r + \alpha) \leqslant (r + \alpha\eta)(r + 2\alpha\eta^2) + 2\alpha\eta^4(\alpha + \alpha\eta) + \alpha\eta^4(r + 2\alpha\eta),$$

which is exactly the same as (5.12) with η written for γ. Hence we similarly get

$$[3]!\, M_3(r)/(r + \alpha) \leqslant (r + 2\alpha\eta)(r + 4\alpha\eta^2),$$

and (5.5) is true for $n = 3$. Supposing that (5.5) is satisfied for $n = 2, 3, 4, \ldots, m$, $m \geqslant 3$ and proceeding in the same way as above, applying (5.2), (5.6), and (5.10), the following inequality can be obtained

$$\begin{aligned}
[m + 1]!\, M_{m+1}(r)/(r + \alpha) &\leqslant (r + \alpha\eta) \prod_{j=1}^{m-1} (r + 2\alpha j\eta^{j+1}) \\
&+ \alpha^2 \binom{m}{1} \eta^{2m}(1 + \eta) \prod_{j=1}^{m-2} (\alpha + 2\alpha j\eta^{j+1}) \\
&+ \alpha^2(1 + \eta) \sum_{k=2}^{m-1} \binom{m}{k} \eta^{2k(m+1-k)} \left\{ \prod_{j=1}^{k-1} (r + 2\alpha j\eta^{j}) \right\} \\
&\times \left\{ \prod_{j=1}^{m-k-1} (\alpha + 2\alpha j\eta^{j+1}) \right\} + \alpha\eta^{2m} \prod_{j=1}^{m-1} (r + 2\alpha j\eta^{j}). \qquad (5.14)
\end{aligned}$$

Now (5.14) is exactly of the same form as (5.13) with η written for γ, and the right-hand sides of both (5.13) and (5.14) are of the same form as that of [1, p. 618, formula (3.6)]. Hence we conclude that

$$[m + 1]!\, M_{m+1}(r)/(r + \alpha) \leqslant \prod_{j=1}^{m} (r + 2\alpha j\gamma^{j}) \qquad (0 < q < 1;\ \beta < 1),$$

and

$$[m + 1]!\, M_{m+1}(r)/(r + \alpha) \leqslant \prod_{j=1}^{m} (r + 2\alpha j\eta^{j}) \qquad (q > 1;\ q\beta < 1).$$

Therefore (5.4) and (5.5) are established and the proof of the theorem is thus complete.

We are now in a position to prove the main Theorem 1.3.

Proof of Theorem 1.3. Suppose that $f(z)$ is any function regular in

$|z| \leqslant r$, where $r \geqslant \alpha$. Then there is a number $r_1 > r$ such that $|f(z)| \leqslant M < \infty$ in $|z| \leqslant r_1$, and consequently we have

$$|f^{(k)}(0)/k!| \leqslant M/r_1^k \qquad (k \geqslant 0). \tag{5.15}$$

Now, since

$$D_q^n f(z) = \sum_{m=0}^{\infty} \frac{f^{(m+n)}(0)}{(m+n)!} x \frac{[m+n]!}{[m]!} z^m, \tag{5.16}$$

then, applying (5.9) for the case $q < 1$, $\beta < 1$ and (5.10) when $q > 1$, $q\beta < 1$, we obtain, from (5.15) and (5.16),

$$\begin{aligned} |D_q^n f(at^n)| &\leqslant \frac{M}{r_1^n(1 - \alpha\beta^n/r_1)} && (q = 0; \beta < 1), \\ &\leqslant \frac{M[n]!\, r_1}{(r_1 - \alpha\beta^n)^{n+1}} && (0 < q < 1; \beta \leqslant 1), \\ &\leqslant \frac{M[n]!\, r_1}{(r_1 - \alpha q^n\beta^n)^{n+1}} && (q > 1; q\beta < 1). \end{aligned} \tag{5.17}$$

Choose the number r' such that

$$r < r' < r_1, \tag{5.18}$$

and the integer n_0 so that

$$r + \alpha\beta^j < r' \qquad r + 2\alpha j\gamma^j < r' \qquad r + 2\alpha j\eta^j < r',$$

for $j \geqslant n_0$, where γ and η are defined in (5.1) and (5.2), respectively. Hence (5.3), (5.4), and (5.5) imply for $n > n_0$, that

$$M_n(r) < Kr'^n/[n]! \qquad (0 \leqslant q < 1 \text{ and } \beta < 1, \text{ or } q > 1 \text{ and } q\beta < 1). \tag{5.19}$$

Finally from (5.17), (5.18), and (5.19) we conclude that the Gonĉarov series

$$\sum_{n=0}^{\infty} D_q^n f(at^n)\, Q_n(z; a, \ldots, at^{n-1}),$$

associated with $f(z)$, will represent $f(z)$ in $|z| \leqslant r$. Theorem 1.3 is therefore established owing to the fact that $\lambda(r) \geqslant \alpha$, $r > 0$.

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